Solution by J. W. NICHOLSON, A. M., LL. D , President and Professor of Mathematics in the Louisiana State University, and Agricultural and Mechanical College, Baton Rouge, Louisiana.

Let x^2, $2(n^2-1)x^2$, $2(n^2+1)x^2$, be the three integers. The first condition gives $4n^2x^4=\square$; hence we have only to solve

$$\left[x^2\right]^3+\left[2(n^2-1)x^2\right]^3+\left[2(n^2+1)x^2\right]^3=\square,$$

$$\text{or } x^6\left[16n^6+48n^2+1\right]=\square,$$

$\therefore\ 16n^6=\left(\frac{48}{2}n^2\right)^2$; whence $n^2=36$; and the three integers are x^2, $70x^2$ and $74x^2$, where x is any integer.

Also solved by *H. W. DRAUGHON, and G. B. M. ZERR.*

24. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.

Solve *generally:* The sum of the cubes of n consecutive numbers is a square. Determine the numbers, when $n=2$, $n=3$, $n=4$, and $n=5$.

I. Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.

Let m, $m+1$, $m+2$, $m+3$, etc., represent any consecutive numbers the sum of whose cubes is to be taken.

Solving by the differential method, we obtain

$$S=\frac{n}{4}\left\{n^3+(4m-2)n^2+(6m^2-6m+1)n+4m^3-6m^2+2m\right\}.$$

This reduces to $S=\frac{1}{4}\left\{[n(n+2m-1)]^2+2m(m-1)[n(n+2m-1)]\right\}$, (A).

If the sum of the consecutive cubes is to be a square, then

$$[n(n+2m-1)]^2+2m(m-1)[n(n+2m-1)]=\square=a^2.$$

Adding $[m(m-1)]^2$ to both members, we have

$$[n(n+2m-1)+m(m-1)]^2=a^2+[m(m-1)]^2.$$

This is of the form $(p^2+q^2)^2=(2pq)^2+(p^2-q^2)^2$.

Equating the respective values, and reducing for m and n, we obtain $2m=1+\sqrt{4p^2-4q^2+1}$, and $2n=\sqrt{4p^2+4q^2+1}-\sqrt{4p^2-4q^2+1}$. It will be observed that each of the radical quantities is of the form of an odd square, $4p+1$.

There are two conditions that will render the radicals rational, and, at the same time, have a an integer:—

(1) When $4q^2=4p$. Then $m=p=q^2$, and $n=1$. According to this condition there is but *one cube* that can be taken at one time, and hence there would be no *sum of cubes.* This cube is the cube of a square, and is, therefore, also the square of a cube.

(2) The second condition is when $p^2=q^2$. Then $m=1$; and substituting this value in (A), we obtain $S=\left\{\frac{n(n+1)}{2}\right\}^2$, which is the square of the sum of the series, $1+2+3+\ldots\ldots+n$. From this, then, we have $1^3+2^3+3^3+\ldots+n^3=(1+2+3+\ldots+n)^2$, or *the square of the sum of the first n natural numbers is equal to the sum of their respective cubes.*

Therefore, in order that the sum of the cubes of n consecutive numbers be a square, ***the first number must be unity.***

When $n=2$, the numbers are 1 and 2; when $n=3$, the numbers are 1, 2, and 3; &c., &c.

Also solved by *Professor COOPER D. SCHMITT, and the PROPOSER.*

II. Solution by B. F. FINKEL, A. M., Professor of Mathematics in Kidder Institute, Kidder, Missouri.

Let $S=1+2+3+\ldots\ldots+n=(n+1)n/2$;

$S_2=1^2+2^2+3^2+\ldots\ldots+n^2=n(n+1)(n+2)/6$; and

$S_3=1^3+2^3+3^3+\ldots\ldots+n^3=?$

$$\begin{aligned}
\text{Now } (n+1)^4-n^4 &= \qquad\qquad\qquad\qquad\qquad 4n^3+6n^2+4n+1\\
n^4-(n-1)^4 &= 4n^3-6n^2+4n-1=4(n-1)^3+6(n-1)^2+4(n-1)+1\\
(n-1)^4-(n-2)^4 &= 4n^3-18n^2+28n-15=4(n-2)^3+6(n-2)^2+4(n-2)+1\\
(n-2)-(n-3)^4 &= 4n^3-30n^2+76n-65=4(n-3)^3+6(n-3)^2+4(n-3)+1\\
\ldots\ldots &= \ldots\ldots\ldots=\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
5^4-4^4 &= \ 369 \ =4\times4^3+6\times4^2+4\times4+1\\
4^4-3^4 &= \ 175 \ =4\times3^3+6\times3^2+4\times3+1\\
3^4-2^4 &= \ 65 \ =4\times2^3+6\times2^2+4\times2+1\\
2^4-1^4 &= \ 15 \ =4\times1^3+6\times1^2+4\times1+1\\
1^4-0^4 &= \ 1 \ =4\times0^3+6\times0^2+4\times0+1
\end{aligned}$$

Adding, $(n+1)^4=4S_3+6S_2+4S+n+1$.

$$\begin{aligned}
\text{Whence, } S_3 &= [(n+1)^4-n-1-6S_2-4S]/4,\\
&= [(n+1)^4-n-n(n+1)(n+2)-2(n+1)n]/4,\\
&= [n^4+2n^3+n^2]/4=[\tfrac{1}{2}n(n+1)]^2.
\end{aligned}$$

If $n=2$, $S_3=9$; if $n=3$, $S_3=36$; if $n=4$, $S_3=100$; if $n=5$, $S_3=225$.

NOTE.—The above method is useful in summing the series: $1^r+2^r+3^r+4^r \cdots\cdots nr$, where r is any integer.

PROBLEMS.

32. Proposed by A. H. BELL, Hillsboro, Illinois.

Decompose into its prime factors the number 549755813889.

33. Proposed by M. A. GRUBER, A. M., War Department, Washington, D. C.

Find three different *sixth powers* whose sum is a square.

[The solution of this problem, if possible, is an answer to the *note* under the solution of Prob. 16.]